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Superconducting flux lattice in a strong magnetic field in the high-T_c superconductors

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Abstract. In a strong magnetic field (flux lattice constant much smaller than penetration depth), vibrations of the vortex lattice are found to decouple from those of the applied field. This suggests that three-dimensional effects (vortex bending) are much more important than in previous theories. A modified Lindemann criterion shows that the vanishing of critical currents in the high-T_c superconductors can be well described by flux lattice melting, with numerical fits yielding estimates for H_{c2} which agree with the most recent previous values. The Lindemann criterion is found to be insensitive to the effective-mass anisotropy of the material except at high fields.

In analysing the dispersion relation for the combined field–vortex oscillations, Doppler-shifted cyclotron resonance of core electrons is found to provide a roton-like minimum.

1. Introduction

A key limitation to technological application of the new high-T_c superconductors [1] has been the very low values of critical current j_c found at temperatures significantly below the critical temperature T_c. The problem is twofold: j_c is low in polycrystalline samples because of weak links between grains, but the intragrain j_c is also strongly temperature and magnetic field dependent, and is small at high T. Recent evidence suggests that this latter problem is due to a 'melting' of the flux lattice in these materials which takes place at relatively low temperatures, particularly in the newer Bi and Tl-based compounds in which the melting occurs near $T_M = 30$–40 K [2, 3].

At present, there is considerable debate as to whether these phenomena represent a true melting of the flux lattice, or whether they are better described in terms of flux creep [4, 5]. In parallel with this, there are questions as to whether the melting is an intrinsic phenomenon, or whether it is extrinsic: severely modified by defects which pin vortices. In light of these severe complications, the goal of the present paper is relatively modest: to examine flux lattice melting in the intrinsic regime, where defects can largely be neglected. Even this goal has not been fully attained: a number of effects—in particular, anisotropy—have not been fully incorporated. Nevertheless, it is hoped that the present results will be of value in better understanding the role of melting in these materials.

In this paper, it is shown that the theoretical elastic response of the flux lattice is drastically modified in a strong magnetic field, B. A new regime of flux lattice vibrations is found (§ 2) when the lattice constant a is much smaller than the magnetic penetration depth λ_s. Here $a^2 = 2\varphi_0/\sqrt{3}B$, $\varphi_0 = hc/2e$ is the flux quantum, $\lambda_s^2 = mc^2/4\pi e\rho_s$, and $\rho_s = n_s e$, with n_s the density of condensed holes. In this regime, the fields of the vortices overlap strongly, so that the magnetic field is nearly uniform spatially, with only a weak modulation due to the vortices. Oscillations of the vortex lattice can decouple from those of the field. The usual elastic constants refer to combined oscillations of the field plus vortex array, and remain finite even for $T > T_c$. New elastic constants are introduced, characteristic of pure flux lattice vibrations. These are considerably softer, and vanish at T_c.

The Lindemann melting criterion, rewritten in terms of the corrected elastic constants, is found to yield extremely low melting fields, and a stronger scaling of field with temperature than that predicted in activated flux creep theories [4, 5].

In the earlier theories of the flux lattice elasticity, the tilt modulus c_{44} was found to be much larger than the shear modulus c_{66}, which was the only elastic constant to vanish at H_{c2}. Under these circumstances, the vortices are stiff rods parallel to the field, and a two-dimensional melting theory [6–8] could be attempted. In the present theory, all lattice constants vanish at H_{c2}, and, $c_{44} \simeq c_{66}$. Hence, the vortices are extremely flexible, and a three-dimensional melting theory appears necessary, perhaps like the vortex ring models of the superfluid transition in ^{4}He. In light of this, it is interesting to note (§ 3) that a weak roton-like minimum occurs in the upper tilting mode (field plus vortex) dispersion relation.

2. Vortex array versus uniform field

The flux lattice is often approximated by an elastic continuum with bulk modulus, c_{11}, shear modulus, c_{66}, and, to account for three-dimensional effects, a tilt modulus, c_{44}. In an anisotropic material—such as a high-T_c superconductor with applied field H parallel to the conducting planes—extra elastic constants are needed, but the above three suffice for a high-T_c material with H perpendicular to the layers.

The field enters the superconductor only via the vortices, and when the vortices are well separated ($a \gg \lambda_s$) the field and vortices must oscillate in unison. However, when $a \ll \lambda_s$, the field is nearly uniform, with the vortex lattice producing a weak periodic modulation of amplitude less than $2H_{c1} \ll H$. For the high-T_c superconductors, $\lambda_s \cong 3000$ Å, and $a = \lambda_s$ at $B = H_{c1} \cong 260$ G. Thus vortices are virtually always in the high field regime. With applied fields greater than 2 T, the flux lattice provides only a very weak modulation. In such a situation, the vortices should be able to decouple from the applied field and vibrate independently. Hence, arguments based on the average c_{11}, c_{44} may be seriously misleading in describing flux lattice melting.

Put another way, the elastic constants describe *combined* vibrations of the flux lattice plus magnetic field. Vibrations with wave vector k parallel to H are predominantly field vibrations, and merge continuously into ordinary helicon waves in the normal metal above T_c. In contrast, transverse modes require an ordered lattice (resistance to shear), and hence vanish above H_{c2} or if the lattice melts—similar to Tkachenko modes in ^{4}He. There is a second, lower-frequency branch of the vibration spectrum with $k \parallel H$, a form of 'second sound' in which the vortices and normal quasiparticles vibrate against one another, but the background field remains unexcited. This branch also vanishes at H_{c2}, and it is this low-frequency branch which is responsible for flux lattice melting. Recognition of this second branch resolves a discrepancy between Brandt's expression [9] for c_{44} and earlier helicon wave calculations, as discussed in § 4.3 below.

It has been suggested to the author that it is not possible to tilt or bend the vortices without similarly perturbing the field B, since 'the field is generated by the vortices'. This statement is demonstrably incorrect, since it would lead to the paradoxical result that the field B must then vanish for $B > H_{c2}$. Instead, the vortices are related to the modulation of the field, and hence to the magnetisation M, where, well above H_{c1}, $4\pi M \ll B$ in the high-temperature superconductors. By restricting the discussion to the regime $a \ll \lambda_s$, the magnetic field can be separated as $B = H + 4\pi M$ where H is equal to the applied field and M is the magnetisation of the vortex lattice. Thus [10]

$$c_{44} = \frac{BH}{4\pi} \tag{1}$$

can be split up as

$$c_{44} = \frac{B^2}{4\pi} - BM = c_{44}^B + c_{44}^v. \tag{2}$$

If an excitation causes *both* the field and the vortices to oscillate, the response is governed by the full c_{44}; if, on the other hand, only the vortices vibrate, then only c_{44}^v is involved. This can be approximated [9] as

$$c_{44}^v = \frac{B(H_{c2} - B)}{8\pi\kappa^2\beta_0} \tag{3}$$

where $\beta_0 = 1.16$, the Ginzburg–Landau parameter, $\kappa = \lambda_s/\xi \geqslant 100$ for the high-T_c materials, and ξ is the superconducting coherence length. In comparison

$$c_{66} \cong \frac{b(H_{c2} - B)^2}{32\pi\kappa^2} \tag{4}$$

where $b = B/H_{c2}$, and equation (4) is valid if $b < 0.25$. Thus for $H_c \ll B \ll H_{c2}$, $c_{66}/c_{44} \cong H_c^2/4BH_{c2} \ll 1$, but $c_{44}^v/c_{66} \cong 4/\beta_0 \cong 1$ (recall that $H_c = \varphi_0/4\pi\lambda_s\xi = H_{c2}/\sqrt{2}\kappa \cong \sqrt{2}\kappa H_{c1}/\ln \kappa$). Associated with the two tilt moduli there will be two oscillation frequencies (calculated as in reference [11]):

$$\omega^h = \left(\frac{4\pi c_{44}}{B}\right)\frac{e}{mc}\frac{k^2\lambda_s^2}{1 + k^2\lambda_s^2} \tag{5a}$$

$$\omega^v = \left(\frac{4\pi c_{44}^v}{B}\right)\frac{e}{mc}k^2\lambda_s^2. \tag{5b}$$

For convenience these two branches are labelled as h(elicon) mode and v(ortex) mode, respectively. A kinetic equation derivation of these results is given in Appendix 1, where it is shown that c_{44}^v corresponds to a second sound vibration, akin to that found in superfluid ^{4}He, and that ω^h (equation 5(a)) must be modified when normal quasiparticles are present. If $B \ll H_{c2}$, then $\omega^v/k^2\lambda_s^2 \cong eH_{c1}/mc \equiv \omega_{c1}$. Thus in the long-wavelength limit $\omega^v/\omega^h \cong H_c^2/\beta_0 BH_{c2} \sim H_{c1}/B \ll 1$. These frequencies are so low that they can easily be thermally excited, leading to significant tilting [12] of the vortex lines, unless they are strongly pinned. Thus if $H_{c1} = 500$ G, $m = 2m_0$, $\omega_{c1} = 0.03$ K. While a vortex lattice may persist, it should not be thought of as being straight, but rather very rubbery. The very different dispersion of the two modes should be noted. The helicon mode saturates at $\omega = \omega_c$ when $k\lambda_s > 1$, since the field cannot respond at wavelengths of less than λ_s. There is no such restriction for the vortex modes and the two branches actually cross at $k\lambda_s \cong B/H_{c1}$, or $ka \cong 1$.

Introduction of the soft tilt modulus c_{44}^{v} modifies the Lindemann melting criterion introduced by Nelson and Seung [13]. They showed that the root-mean-square fluctuations of the vortex line displacement could be written as

$$\bar{u}^2 \cong \left(\frac{n}{4\pi c_{44} c_{66}}\right)^{1/2} k_B T \tag{6}$$

where $n = B/\varphi_0$. According to the Lindemann criterion, the lattice should melt when $\bar{u}$ is about 10% of the lattice constant a. Since the vortices contribute to two normal modes of the coupled system, 'ordinary' and 'second' sound, the root-mean-square fluctuation should be written as a weighted sum of contributions from both modes. However, since the vortex mode is considerably softer, only it will be considered in the present paper. Making the substitution $c_{44} \to c_{44}^{v}$, equation (6) can be written as

$$\frac{\bar{u}^2}{a^2} = \left(\frac{48\pi\kappa^4\beta_0 B}{\varphi_0^3 H_{c2}^2}\right)^{1/2} \frac{k_B T}{(1-b)^{3/2}}. \tag{7}$$

Equation (7) has two interesting features. First, it predicts a very low value of the melting temperature. Thus, using values for the Bi compounds ($\kappa \cong 200$, $H_c \cong 2$ T at 77 K), $\bar{u} = 0.1a$ at $B = 11$ G—that is the lattice is essentially melted at H_{c1}. Moreover, equation (7) has an interesting scaling form: if $\bar{u}/a$ has a constant value and $b \ll 1$, then

$$B \approx H_{c2}^2/T^2 \sim (1 - T/T_c)^2/T^2. \tag{8}$$

Since flux creep theory predicts $H \sim (1 - T/T_c)^{3/2}/T$, this suggests that flux creep will dominate near T_c, and flux lattice melting at lower T. A comparison with experiment [3, 14, 15] will be presented in § 4.

In an anisotropic superconductor, it should be noted that c_{44}^{v} is itself highly anisotropic, via H_{c2} and κ, whereas c_{44} is nearly isotropic, being dominated by the field effects. Some care must be taken in generalising equation (7) to an anisotropic system. This is further discussed in § 4.4 and Appendix 2.

3. Magnetorotons (?)

At H_{c2}, the lower vibration mode ω^{v} vanishes. The upper mode should extrapolate to the normal-state helicon mode. Equation 5(a) does not, and it is instructive to trace the source of this error. A helicon wave in a normal metal can excite a Doppler-shifted cyclotron resonance [16]. The resonance condition $\omega + kv_F \cong \omega_c$ becomes approximately

$$kr_c = 1 \tag{9}$$

where $r_c = v_F/\omega_c$ is the cyclotron radius, since $\omega \ll \omega_c$. This produces a form of Landau damping (strictly, cyclotron damping) of the wave: an electron propagating parallel to B can 'surf ride' and resonantly absorb energy from the helicon wave. In Appendix 1.3, it is shown that similar phenomena occur in a superconductor, with excitation of the

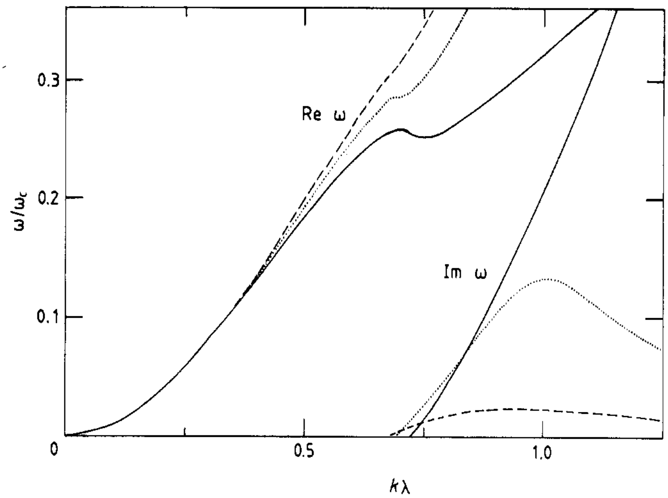

Figure 1. Normalised dispersion relations ω/ω_c plotted against $k\lambda$ for helicon waves in a superconductor, assuming $B = 0.1$ T, $\rho_n/\rho = 0.1$ (broken curves), 0.4 (dotted curves), 1.0 (full curves).

'normal' electrons in the vortex cores. The resulting dispersion of the helicon wave is equivalent to equation 5(a), with a renormalised penetration depth:

$$\lambda^2 \rightarrow \lambda^2/[1 + \rho_n\rho^{-1}(\Gamma - 1)]. \tag{10}$$

Here ρ_n is the normal electron (quasiparticle) charge density, $\rho = \rho_s + \rho_n$, $\lambda^2\rho = \lambda_s^2\rho_s$, and Γ is defined in Appendix 1.

Numerical estimates of the dispersion curve parameters are hampered by lack of detailed knowledge of the hole band parameters. Assuming the carriers form a single band with only spin degeneracy, $n = 3 \times 10^{21}$ cm^{-3} and $m^*/m_0 = 2$ (= ratio of effective mass to free-electron mass), then $\hbar\omega_c/k_B \cong 0.67\,B_T$, where B_T is the applied field in teslas, and $v_{F\parallel} = 2.7 \times 10^7$ cm s^{-1} is the in-plane Fermi velocity. The relevant cyclotron radius involves $v_{F\perp}$, which is about 100 times smaller, so $r_c = v_{F\perp}/\omega_c \cong 310$ Å$/B_T$. The 'roton' dip occurs when $\omega_c = \omega + kv_{F\perp}$, or, for $B > 1$ T, at approximately $\omega = \omega_c$, $k = k_c \cong 1/(\lambda_s^2 r_c)^{1/3} \cong B_T^{1/3}/1400$ Å, or $k_c a \approx 1/(2.9B_T^{1/6})$. Using these parameters, figures 1–3 plot the helicon dispersion curves for a variety of values of B and ρ_n/ρ; a change in curvature can still be seen for $\rho_n/\rho = 0.1$. In all cases, the threshold of the Doppler-shifted cyclotron resonance is accompanied by the onset of Landau damping, so the helicon waves are poorly defined at larger κ-values. It is expected that $\rho_n/\rho \cong B/H_{c2}$, so that at low T this factor can be large for fields parallel to the c-axis.

When $\omega\tau > 1$, the surf-riding phenomenon can lead to the generation of solitons in a plasma [17]. A similar phenomenon should arise in a superconductor, creating a localised excitation along the vortex core which has the properties of an incipient vortex ring. However, it is not clear if this occurs in the high-T_c superconductors. The normal state $\tau^{-1} \propto T$. If this same power law continues to hold for $T < T_c$, the scattering rate will be too large for the solitons to exist as well-defined modes, except at very low temperatures. Since the mechanism producing the linear-in-T scattering is not well understood, it remains possible that τ will be cut off below some $T < T_c$.

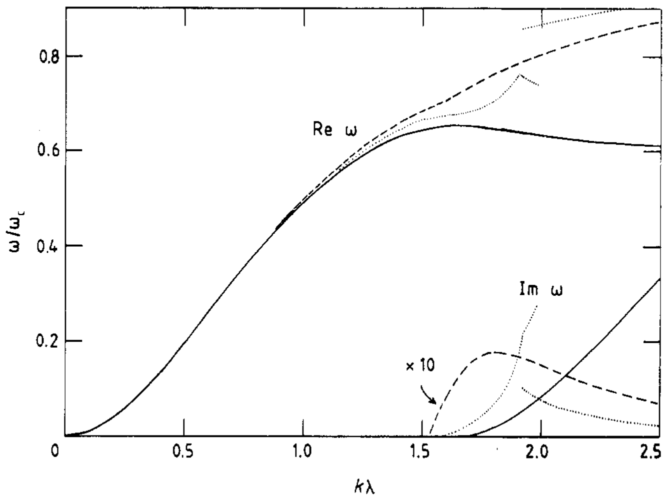

Figure 2. Helicon dispersion relation, as in figure 1, for $B = 0.5$ T. Note that two solutions appear over a limited range of $k\lambda$ for $\rho_n/\rho = 0.4$.

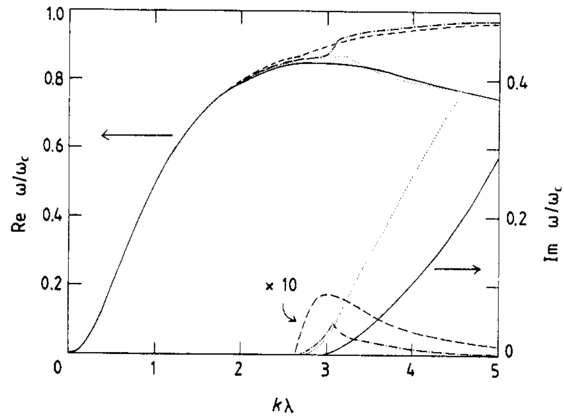

Figure 3. Helicon dispersion relation, as in figure 1, for $B = 2.0$ T, with an additional curve $\rho_n/\rho = 0.3$ (the chain curve).

The above discussion has dealt with the helicon branch of the dispersion relation, equation 5(a). Since the vortex waves, equation 5(b), couple more strongly to normal quasiparticles, it might be thought that they would show a similar roton dip. However, it can be seen from the equations of Appendix 1 that the normal electron conductivity, σ_{xy}, does not contribute to this dispersion relation. It may be that a dip occurs near $ka = 1$, where the two branches of the dispersion curve cross.

4. Discussion

4.1. The Lindemann criterion

In a weak-pinning superconductor (fewer pins than vortices), it is the vortex lattice's resistance to shearing which prevents unpinned vortices from moving in response to an

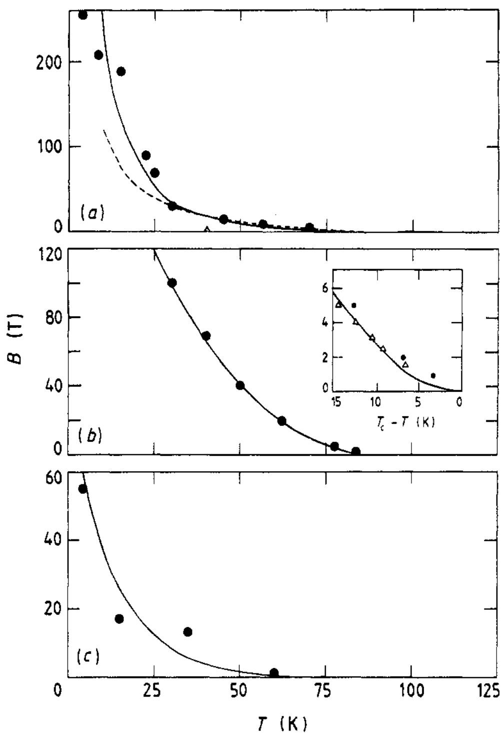

Figure 4. Vortex lattice melting phase diagram, determined by vanishing of j_c (full circles) or mechanical measurements (triangles—from [2]). Full curves correspond to the Lindemann criterion, equation (7), $\bar{u} \rightarrow 0.1\,a$, with parameters listed in table 1; broken curve, flux creep theory. Part (*a*), $TlBa_2Ca_3Cu_4O_x$, data of [3]; (*b*), $YBa_2Cu_3O_{7-\delta}$, [14]; (*c*), $TlBa_2CaCu_2O_x$, [15].

applied current. Hence, when the flux lattice melts, the critical current j_c should fall to zero. While the vanishing of j_c would offer an extremely convenient measure of flux lattice melting, its interpretation can be ambiguous, and has also been interpreted in terms of the flux creep [4, 5]. In attempting to assess the role of flux lattice melting, the present section compares the experimental observations with the pure melting theory predictions. If it is assumed that the field H_M at which j_c vanishes corresponds to flux lattice melting, then the Lindemann criterion can be compared with experiment [3, 14, 15]. Now, j_c is found to scale with field in a manner similar to that found by Kramer [18] in conventional superconductors (Hettinger *et al* [14] have shown that a very similar scaling is found in flux creep theory). By using this scaling, values of H_M can be extrapolated to fields considerably higher than can be measured directly. Figure 4 plots the resulting values of H_M for three different superconductors, and compares them with the Lindemann criterion prediction, equation (7), with $\bar{u} = 0.1a$. To minimise the number of parameters, κ has been taken to be T-independent, while $H_{c2}(T)$ was assumed to have the same form as in reference [3], but scaled to $H_{c2}(0)$. The fit parameters are listed in table 1, and compared with other measurements of H_{c2}, employing single crystals [19–22]. Most of these measurements underestimate H_{c2}, since they have assumed that this is the field at which the resistance has fallen to half its normal state value. It is now recognised that the resistive transition is greatly broadened by 'flux creep', and that H_{c2}

Table 1. Estimates of $H_{c2}(0)$ and κ.

Material[a]	$H_{c2}(0)$ (T) $\parallel$	$\perp$	κ $\parallel$	$\perp$	Method	References
Eu 123	390	34	400[b]	35[c]	R 50[d]	[19]
Y 123	510	118	390[b]	90[c]	R 50	[20]
Y 123	1270	195	750[b]	115[c]	R90	[21]
Y 123		300		150	Fig. 4(*b*)	[14]
Bi 2212	1550	31	2500[b]	50[c]	R 3	[22]
Bi 2212	2640	44	3300[b]	55[c]	R 50	[20]
Tl 1212	100		200		Fig. 4(*c*)	[15]
Tl 1234	1900		700		Fig. 4(*a*)	[3]

[a] Materials are abbreviated as Tl 1234 = $TlBa_2Ca_3Cu_4O_x$, Y 123 = $YBa_2Cu_3O_{7-\delta}$, etc.
[b] Estimated as $\kappa = \lambda/\xi_c$, with ξ_c derived from H_{c2} and λ assumed to be $\cong$1500 Å, appropriate to $YBa_2Cu_3O_{7-\delta}$.
[c] Assumes $\kappa_\parallel/\kappa_\perp = H_{c2\parallel}/H_{c2\perp}$.
[d] R $X\%$ means that H_{c2} is estimated as the field at which the resistance drops to $X\%$ of its normal state value.

probably lies closer to the field of 90% normal state resistance. As can be seen from the table, this field is considerably higher.

Figure 4 shows that this simple melting theory provides a remarkably good description of the vanishing of j_c. Moreover, the values of H_{c2} and κ (table 1) are quite close to the most recent estimates for these values. Nevertheless, the following points should be noted:

(i) The theory seems to underestimate the melting field in the higher temperature regime. This is probably because flux creep becomes important near T_c, as suggested in reference [3b]. The broken curve in figure 4(*a*) shows that the data follow the scaling relation expected from flux creep theory in the higher temperature regime. A clear crossover from flux creep to a melting regime occurs near 40 K. Gammel *et al* [2] by studying the coupling of the flux lattice to mechanical vibrations of the sample, have shown that even near zero field, the onset of melting in Tl- and Bi-based high-T_c superconductors occurs near 30–40 K (triangles in figure 4). This result does not follow from the simple Lindemann criterion, which always predicts melting at T_c as $B \to 0$ (although the presence of rotons might change this). The observed lowering of the zero-field melting temperature below T_c could be due to pinning [3].

(ii) The theory provides a simple explanation for the striking difference between the Tl- (and Bi-) based compounds, which melt near 30–40 K, and $YBa_2Cu_3O_{7-\delta}$, which only melts near T_c. Indeed $TlBa_2Ca_3Cu_4O_x$ and $YBa_2Cu_3O_{7-\delta}$ have nearly the same value of H_c (1.9 and 1.4 T respectively), so the difference is due solely to the much larger Ginzburg–Landau parameter ($\kappa \cong 700$) of the former compound. However, before this conclusion can be firmly accepted, the effects of anisotropy on the Lindemann criterion should be fully worked out. In particular, it is very curious that experimentally [2, 3] there appears to be relatively little dependence of H_M on the orientation of the field.

4.2. *Role of pinning and flux creep*

The above calculations have entirely neglected the role of pinning (except to provide a finite j_c). It is known [23] that collective pinning effects introduce finite correlation lengths both parallel and perpendicular to the applied field. These correlation lengths have been interpreted as the average spacing between dislocations in the vortex lattice [24]. Hence, the flux lattice melting must be modified near the melting transition, when the flux lattice correlation length becomes comparable with the dislocation spacing. However, if the pinning is weak, this dislocation spacing is large, and these finite-size corrections to melting theory can be treated perturbatively [3a]. Since the impurity correlation lengths scale to zero at H_{c2}, corrections to melting theory will always be important at sufficiently high fields, but still may be small in the Tl and Bi compounds, where $H_M \ll H_{c2}$.

Experimentally, it has proven difficult to assess the importance of pinning. The fact that the melting temperature is similar in both Bi- and Tl-based superconductors, whether single crystal [2] or polycrystalline [3] suggests that pinning effects might be weak, and the melting an intrinsic effect. Pinning effects should probably be stronger in $YBa_2Cu_3O_{7-\delta}$, where the twin boundaries act as pinning centres. However, the melting temperature is considerably higher in this material, the opposite of the effect expected from pinning.

The flux creep mechanism requires a flux lattice with many dislocations in it—in the absence of dislocations, any collective motion of a vortex bundle would generate vacancies and interstitials in the flux lattice, and the high energy cost of these (involving lattice compression) is neglected in theoretical treatments of flux creep. Hence, in the weak pinning regime, flux creep can probably be neglected.

Any intrinsic melting theory seems to predict that the melting temperature must extrapolate to T_c as $B \rightarrow 0$, in disagreement with experiment. In reference [3a], it was shown that in a two-dimensional melting theory (large c_{44}), pinning acts to reduce T_M below T_c, but that very large levels of pinning are required to obtain agreement with experiment. In the present three-dimensional theory, the soft c_{44} greatly reduces T_M at modest magnetic fields, and suggests that a relatively small pinning density could cause $T_M(H = 0) \ll T_c$. Alternatively, the reduction of T_M could be due to the second (ordinary sound) branch of lattice vibrations, in particular the thermal excitation of magnetorotons.

4.3. *Other calculations*

After this paper was submitted for publication, I became aware of several related calculations [25–27] using the Lindemann criterion to describe flux lattice melting. Moore's [25] treatment is similar to the present one. He also finds two branches of the dispersion relation. His Lindemann criterion differs from the present result only in that he used an expression for c_{66} which is valid in the high-field ($b > 0.4$) limit. References [26–27] both use Brandt's Ginzburg–Landau expression [9] $c_{44}(k)$. The resulting Lindemann melting criteria are virtually identical to equation (7), except for an overall multiplicative constant. In particular, the scaling relation, equation (8), is unchanged.

It is important to clarify the relation between the Ginzburg–Landau expression [9] for $c_{44}(k)$ and the present two-fluid calculation. Brandt calculated $c_{44}(k)$ by a 'frozen phonon' technique—fixing the vortex to have a particular k-value, then calculating the Ginzburg–Landau energy of that configuration. The resulting $c_{44}(k)$ had a similar dispersion to that found early by de Gennes and Matricon [11, 28], but differed significantly at high fields and temperatures. If it were used to calculate the helicon dispersion relation, it would result in an expression similar to equation 5(a), but with a modified λ_s, which diverges both at T_c and at H_{c2}. This would result in a discontinuity in the helicon dispersion relation at H_{c2}, which is not observed experimentally [29]. The present two-fluid calculation explains the reason for this disagreement. A pure vortex oscillation is actually a linear combination of several of the normal modes of oscillation of the combined quantum fluid. The resulting c_{44} is dominated by the helicon mode at low q and by the second sound mode at high q. The vortex displacement vector should be a weighted sum over the normal mode components, but is dominated by the lightest mode. The difference between the present equation (7) and the results of references [26, 27] is presumably the weight factor of the second sound mode. However, the present calculation is considerably more amenable to incorporation of anisotropy effects, as shown in § 4.4.

A word of caution to experimentalists. The present calculation of the Lindemann criterion depends on energetics only, and is insensitive to dynamics. Thus the calculation holds even for the high-T_c superconductors, where both the helicons and second sound modes are probably overdamped, due to the large normal state resistivity. The best chance of seeing these effects would be in a conventional superconductor, such as Nb, where helicon waves have already been observed [29].

4.4. Anisotropy

Combining the work of Nelson and Seung [13] with the recent work of Kogan and Campbell [30] on c_{66} anisotropy, it is possible to calculate the way in which the strong effective-mass anisotropy modifies the Lindemann criterion, equation (7). Anisotropy has two effects: first, H_{c2} must be replaced by its proper angular-dependent value; then the Ginzburg–Landau parameter must be replaced by an effective, angle-dependent value, κ_{eff}. Explicit calculations (Appendix 2) show $\kappa_{eff\perp} = (\kappa_\perp \kappa_\|)^{1/2}$, while $\kappa_{eff\|} = \kappa_\|$. Hence, the combination κ_{eff}^4/H_{c2}^2, occurring in equation (7), is *the same* for these two extreme cases, strongly suggesting that it is independent of angle. In other words, the Lindemann melting criterion is independent of field direction (except for the explicit factor, $1 - b$).

This calculation confirms a conjecture by Bishop [31], that the effect of a soft c_{44} for B perpendicular to the (a, b) plane exactly compensates the effect of an extremely small c_{66} along the easy axis (perpendicular to c) when B lies in the (a, b) plane. It further explains the experimental results that the melting field is almost independent of angle. To correct table 1 for anisotropy, the only change is that the measured $\kappa_{eff} = 150$ for $YBa_2Cu_3O_{7-\delta}$ should be compared with $(\kappa_\perp \kappa_\|)^{1/2} = 120$ [19], 190 [20], or 290 [21].

4.5. Rotons

The Lindemann criterion is not in itself a theory of melting. The low bending energy (§ 2) shows that vortices are far from straight, suggesting that two-dimensional theories are unlikely to describe the transition correctly, and that flux entanglement effects [12] are important. Just as in ^{4}He, thermally excited vortex rings may provide the key to the transition. Such rings could allow a reconnection of adjacent vortex lines, allowing the array to slip gradually by pinning centres. Whether the roton-like minimum discussed in § 3 is relevant to this transition can only be determined by more detailed investigation, including a better understanding of the role of viscosity and dissipation. However, this roton may be important in other systems, and its relation to the roton mode in superfluid ^{4}He should be explored.

Acknowledgments

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Appendix 1. Kinetic equations for a superconductor

A1.1. Usual dispersion relations, local limit

Neglecting pinning and dissipative terms, the linearised kinetic equations for a superconductor can be written in the local limit as

$$\frac{m}{e}\frac{\partial \boldsymbol{v}_\mathrm{n}}{\partial t} - \frac{\boldsymbol{v}_\mathrm{n} \times \boldsymbol{B}}{c} = \boldsymbol{E} + \frac{\rho_\mathrm{s}}{\rho_\mathrm{n}} \nabla \mu \tag{A1.1}$$

$$\frac{m}{e}\frac{\partial \boldsymbol{v}_\mathrm{s}}{\partial_t} - \frac{\dot{\boldsymbol{\varepsilon}} \times \boldsymbol{B}}{c} = \boldsymbol{E} - \nabla \mu \tag{A1.2}$$

$$\frac{m}{e}\frac{\partial \boldsymbol{v}_\mathrm{s}}{\partial t} - \frac{\boldsymbol{v}_\mathrm{s} \times \boldsymbol{B}}{c} = \boldsymbol{E} + \frac{B}{c} \nu \frac{\partial^2 \varepsilon}{\partial z^2} - \nabla \mu \tag{A1.3}$$

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{4\pi}{c^2}\frac{\partial j}{\partial t} \tag{A1.4}$$

$$j = \rho_\mathrm{n} \boldsymbol{v}_\mathrm{n} + \rho_\mathrm{s} \boldsymbol{v}_\mathrm{s} \tag{A1.5}$$

where $v_{\mathrm{n(s)}}$ is the normal fluid (superfluid) velocity, $\rho_{\mathrm{n(s)}}$ the corresponding charge density, $\boldsymbol{\varepsilon}$ the vortex array displacement vector, μ is the chemical potential, and $\nu = (\hbar/4m)\ln \kappa$. Equation (A1.3) has been simplified by assuming that $\boldsymbol{\varepsilon}$ is a function of z (the field direction) and t only. These equations are actually based on the equations for vortices in ^{4}He [32] using the well-known correspondences between the two systems [33]. All vectors are assumed to have the form of plane waves polarised in the x–y plane, propagating along z; for example, $\boldsymbol{E} = \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}(kz-\omega t)}$, etc. From these equations, the helicon-like dispersion can be derived. For ordinary helicon waves $\nabla \mu = 0$. The resulting dispersion relation is complicated, but simplifies in some limiting cases:

(i) for $\rho_\mathrm{n} = 0$:

$$\omega = \omega_\mathrm{c} \frac{k^2 \lambda^2}{1 + k^2 \lambda^2} + \frac{\hbar k^2}{4m} \ln \kappa. \tag{A1.6}$$

The first term on the right-hand side is the usual result due to interacting vortices, while the second term is the oscillation frequency of a single vortex. It may be rewritten in the suggestive form $\omega_{\mathrm{c1}} k^2 \lambda_\mathrm{s}^2$ where

$$\omega_{\mathrm{c1}} = eH_{\mathrm{c1}}/mc \tag{A1.7}$$

and H_{c1} is the lower critical field. Hence, the second term is negligible in the usual case $B \gg H_{\mathrm{c1}}$.

(ii) For the normal state, $\boldsymbol{v}_s$, $\boldsymbol{\varepsilon}$ vanish, $\rho_s = 0$, and

$$\omega = \omega_c \frac{k^2\lambda^2}{1 + k^2\lambda^2} \tag{A1.8}$$

where now $\lambda^2 = mc^2/4\pi e\rho_n$.

(iii) Finally, if both ρ_s and ρ_n are non-zero, but $\omega \ll \omega_c$, and ν is neglected, the dispersion relation is identical to equation (A1.8), but with $\lambda^2 = mc^2/4\pi e\rho$, $\rho = \rho_n + \rho_s$. With this definition of λ, and neglect of the ν-term, equation (A1.8) holds in all three regimes (i)–(iii). For comparison with the second sound mode in Appendix 1.2, the solution of equations (A1.1)–(A1.5) in this case will now be discussed in more detail. Setting $\boldsymbol{\nabla}\mu = 0$ and neglecting ε in equation (A1.3), the dispersion is found from the simultaneous solution of equations (A1.4) and

$$\frac{m}{e}\frac{\partial j}{\partial t} - \frac{j \times \boldsymbol{B}}{c} = \rho \boldsymbol{E} \tag{A1.9}$$

which follows from (A1.1) and (A1.3). Assuming that both $\boldsymbol{j}$ and $\boldsymbol{E}$ vary as $\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}}\mathrm{e}^{\mathrm{i}(kz-\omega t)}$ then yields equation (A1.8) and the ratio $j/E = \mathrm{i}ck^2/4\pi\omega$. As is usual for helicon waves, only one of the two modes $\hat{x} \pm \mathrm{i}\hat{y}$ is propagating, depending on the sign of the charge carriers.

Appendix 1.2. Second sound

Equations (A1.1)–(A1.5) also have a second-sound solution. Recall that in superfluid helium second sound is a coupled vibration in which the normal and superfluid carriers vibrate out of phase with one another (as in an optical mode), but in such a way that the total density $\rho_s + \rho_n$ is constant. Since there is therefore no long-range Coulomb forces, this mode goes to zero frequency at $k = 0$, as in a sound mode. The energy, however, is transported collectively, in a temperature oscillation.

Similarly, in equations (A1.1)–(A1.5), there is an independent propagating solution associated with a temperature wave $\boldsymbol{\nabla}\mu = S\boldsymbol{\nabla}T$ (S is the entropy), which is decoupled from low-frequency magnetic field vibrations ($\boldsymbol{v}_s = 0$). It can be understood as a coupled vibration of the vortices (ε) and the normal fluid ($\boldsymbol{v}_n$), and vanishes as $\rho_n \rightarrow 0$.

Explicitly, let $\boldsymbol{\nabla}\mu$, $\boldsymbol{v}_n$, $\boldsymbol{E}$ and $\boldsymbol{\varepsilon}$ vary as $(\hat{\boldsymbol{x}} \pm \mathrm{i}\hat{\boldsymbol{y}})\mathrm{e}^{\mathrm{i}(kz-\omega t)}$. Then the amplitudes are related by

$$\rho_n v_n = -\mathrm{i}k^2c^2E/4\pi\omega \tag{A1.10}$$

$$E = -\rho_s|\boldsymbol{\nabla}\mu|\left[\rho_n + \frac{m_c^2k^2}{4\pi e}\left(1 \pm \frac{\omega_c}{\omega}\right)\right]^{-1} \tag{A1.11}$$

$$|\boldsymbol{\nabla}\mu| - E = -\frac{B\nu k^2}{c}\varepsilon \tag{A1.12}$$

while ε satisfies the equation

$$\frac{\dot{\boldsymbol{\varepsilon}} \times \boldsymbol{B}}{c} = -\frac{B}{c}\nu\frac{\partial^2\boldsymbol{\varepsilon}}{\partial z^2} \tag{A1.13}$$

with the dispersion relation

$$\omega = \pm\nu k^2 \tag{A1.14}$$

(again, only one mode is propagating).

This mode is essentially the same as equation (5*b*). Thus $\nu = \omega_{c1}\lambda_s^2$, and the dispersion relations would be identical if $4\pi c_{44}^{v}/B = H_{c1}$. Instead, from equation (3), $4\pi c_{44}^{v}/B = \gamma H_{c1}(1-b)$, with $\gamma = 1/\beta_0 \ln(\kappa)$. These differences are superficial. The parameter γ is of the order one, and is typical of differences between two-fluid and Ginzburg–Landau calculations. (Such theories often approximate $H_{c1} \sim H_c/\kappa$, without the factor $\ln \kappa$.) Finally, the factor $1-b$ is absent because the kinetic equations do not treat the critical regime properly: c_{44}^{v}, like c_{66}, must vanish at H_{c2} when the superconducting phase is no longer present. Thus, I propose identifying the vortex oscillations (5*b*) with the second-sound mode herein derived, equation (A1.14).

Appendix 1.3. Non-local corrections: Doppler-shifted cyclotron resonance

Equation (A1.1) must be modified in the normal state to account for non-local effects. In the low-frequency regime, (A1.1) becomes

$$v_n = \sigma_{xy}(k)E/\rho_n \tag{A1.15}$$

where $\sigma_{xy}(k) \to \rho_n e/m\omega_c$ as $k \to 0$. The dispersion of σ_{xy} arises because of cyclotron resonance, which becomes Doppler shifted for a finite-wavelength disturbance, $\omega_c = \omega + kv_F \to kv_F$, where v_F is the Fermi velocity of normal electrons. The resulting conductivity is [16, 34]

$$\sigma_{xy} = \frac{\rho_n e}{m\omega_c}\Gamma \tag{A1.16}$$

$$\Gamma = \tfrac{3}{4}\eta\left[(1-\eta^2)\ln\left(\left|\frac{n+1}{\eta-1}\right|\right) + 2\eta - \pi i(1-\eta^2)\theta(1-\eta)\right] \tag{A1.17}$$

$$\eta = (\omega_c - \omega)/kv_F = (1-\omega/\omega_c)/kr_c \tag{A1.18}$$

where θ is a step function, $\theta(x) = 1(0)$ if $x > (<)0$.

This in turn modifies the dispersion relation (A1.8) to

$$\omega = \frac{k^2c^2}{4\pi\sigma_{xy}} \tag{A1.19}$$

which develops an imaginary part (cyclotron damping) for $kr_c > 1$, with a kink in the real part.

In the superconducting state, similar non-local effects should arise for the vortex cores, at least for propagation parallel to the vortex axis. Cyclotron-damping effects are unlikely to involve superfluid electrons, because of the gap in their excitation spectrum. Assuming only the normal electrons are affected, the dispersion relation in the superconducting state becomes

$$\omega = \frac{\omega_c k^2\lambda^2}{1 + \rho_n\rho^{-1}(\Gamma - 1) + k^2\lambda^2}. \tag{A1.20}$$

This form correctly interpolates between the normal state result, (A1.19), and the $\rho_n \to 0$ result. In the high-field, low-T regime, $\rho_n/\rho \to B/H_{c2}$.

Appendix 2. The Lindemann criterion in an anisotropic superconductor

Neglecting the small *a–b* plane mass anisotropy, the high-T_c superconductors can be represented as uniaxially anisotropic materials, $m_a = m_b = m_1$, $m_c = m_3$ with $m_1/m_3 \equiv \gamma^4$, in the notation of Kogan and Campbell [30]. The Lindemann criterion now depends on the angle ϕ between the B-field and the c-axis because c_4 and c_{66} (in equation (6)) are angle dependent. There are two modifications to equation (7): first H_{c2} should be replaced by $H_{c2}(\phi)$; secondly κ must be replaced by $\kappa_{\rm eff}(\phi)$. In this appendix, I demonstrate that $\kappa_{\rm eff}^4/H_{c2}^2$ has the same value in the two extreme cases, $\phi = 0, \pi/2$. This strongly suggests that the Lindemann melting criterion is angle independent (except at very high fields, where the factor $1 - b$ becomes important).

For $\phi = 0$, $c_{66\perp}$ has the form of equation 4, with the parameters H_{c2} and κ taking on their appropriate ($\perp$) values. For $c_{44\perp}^{\rm v}$, the expression, equation (3) with $\perp$ values must be multiplied by γ^4, as in the Nelson–Seung conjecture [13] (although c_{44}^B is independent of ϕ—see reference [27]). This yields $\kappa_{\rm eff}^2(0) = \kappa_\perp \kappa_\parallel$, since $\kappa_\perp/\kappa_\parallel = H_{c2\perp}/H_{c2\parallel} = \gamma^2$.

The case where B is parallel to the $a - b$ plane is considerably more complicated, since c_{44} and c_{66} both depend on the angle θ between the c-axis and the component of flux-line motion perpendicular to B. Following Nelson and Seung [13],

$$c_{44\parallel}^{\rm v}(\theta) = \frac{c_{44\perp}^{\rm v}}{\gamma^6}(\gamma^4\cos^2\theta + \sin^2\theta) \qquad \text{(A2.1)}$$

while, using the calculation of Kogan and Campbell [30]

$$c_{66\parallel}(\theta) = \frac{c_{66\perp}}{\gamma^2}(\gamma^8\cos^2\theta + \sin^2\theta - \Gamma\sin^2\theta\cos^2\theta) \qquad \text{(A2.2)}$$

with $\Gamma = (1-\gamma^4)^2$ (in Kogan and Campbell's notation, $\Gamma = 2(C_{66}^{(e)} + C_{66}^{(h)} - 2C_{\rm sq} - 2\eta/C_{66}^{(h)})$ equation (6) must then be replaced by its average over θ. When this is done (numerically), it is found that equation (7) holds with $\kappa_{\rm eff\parallel} = \kappa_\parallel$. Thus for $\phi = \pi/2$

$$\kappa_{\rm eff\parallel}^4/H_{c2\parallel}^2 = \kappa_\parallel^4/H_{c2\parallel}^2 = \kappa_\parallel^2\kappa_\perp^2/H_{c2\perp}^2 = \kappa_{\rm eff\perp}^4/H_{c2\perp}^2.$$

Thus, as claimed, the only angular dependence in equation (7) comes from the explicit factor b.

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